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PERIODIC AND CONDITIONALLY PERIODIC SOLUTIONS IN THE PROBLEM OF MOTION OF A HEAVY SOLID ABOUT A FIXED POINT*

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Several sets of periodic and conditionally periodic solutions of the problem of motion of a solid about a fixed point in a homogeneous gravitational field are investigated. The theory of periodic solutions of Poincaré for Hamiltonian systems of standard form are used for proving the existence of such solutions, for analyzing stability and deriving basic terms of their representative series. The distribution of mass in the body is assumed to be close to axisymmetric and the fixed point to be near the center of mass.

The existence of periodic solutions in the problem of motion of a solid with a fixed point proved in /1/, where the Poincaré method was used for obtaining two sets of periodic solutions, generated by the respective Euler's periodic solutions, for axisymmetric and non-symmetric solids. Existence of similar solutions was earlier indicated by the authors of /2/.

1. Consider the motion of a solid body about a fixed point O in a homogeneous gravitational field. Let $Oxyz$ be a fixed coordinate system with origin at the body fixed point O and axis Oz directed vertially upward, $O\xi\eta\zeta$ be a system of coordinates whose axes coincide with the body principal axes of inertia about O. We introduce two intermediate planes Q_1 and Q_2 passing through the fixed point, the first orthogonal to vector $\mathbf{G}$ of the rotary motion moment of momentum and the second orthogonal to segment OC joining the fixed point and the body center of mass C.

The position of the body center of mass is determined in its "proper" coordinate axes by the constant coordinates r, φ, λ, where λ is the angle between axis $O\xi$ and the line of intersection of planes $O\xi\eta$ and Q_2, φ is the angle between the line OC and the axis of inertia $O\zeta$, and r is the length of OC.

We define the body rotary motion about the fixed point by the canonical Andoyer variables L, G, H, l, g, h /2/, in which the kinetic energy T and the force function of the problem are of the form

$$T = \frac{G^2 - L^2}{2}\left(\frac{\cos^2 l}{B} + \frac{\sin^2 l}{A}\right) + \frac{L^2}{2C}$$
$$U = mrRn^2[(\alpha \sin\lambda - \beta\cos\lambda)\sin\varphi + \gamma\cos\varphi]$$

where A, B, C are the body principal moments of inertia, n is a constant coefficient equal to the angular velocity of motion on a circular orbit of radius R (close to the planet surface), $n^2 = fm_0/R^3$, f is the gravitational constant, m_0 and m are, respectively, masses of the planet and of the solid body, R is the radius of the planet generating the gravity field, and α, β, γ are directional cosines of the gravity force vector in the moving axes $O\xi\eta\zeta$

$$\alpha = \sin\rho\,(\cos l \sin g + \sin l \cos g \cos\theta) + \cos\rho \sin l \sin\theta$$
$$\beta = -\sin\rho\,(\sin l \sin g - \cos l \cos g \cos\theta) + \cos\rho\cos l \sin\theta$$
$$\gamma = -\sin\rho\sin\theta\cos g + \cos\theta\cos\rho$$
$$(\cos\theta = L/G, \quad \cos\rho = H/G)$$

where θ is the angle between the body axis of inertia $O\zeta$ and vector $\mathbf{G}$, ρ is the angle between the fixed (vertical) axis Oz and vector $\mathbf{G}$.

We pass, for convenience, to new canonical variables in conformity with one of the formulas

$$L' = \frac{L}{A\omega_i}, \quad G' = \frac{G}{A\omega_i}, \quad H' = \frac{H}{A\omega_i}, \quad l' = l, \quad g' = g,$$
$$h' = h, \quad \tau = \omega_i t, \quad F' = \frac{F}{A\omega_i^2} \quad (i = 1, 2)$$

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*Prikl.Matem.Mekhan.,45,No.1,535-544,1981 where τ is the new independent variable, and F' is the transformed Hamiltonian. In the first case $\omega_1 = n$, and in the second $\omega_2 = n_2^{(0)}$, where $n_2^{(0)}$ is the unperturbed angular velocity of rotation of the body $n_2^{(0)} = G_0/A$ (G_0 is the respective moment of momentum).

Omitting for simplicity the primes at new variables, we represent the equations or rotary motion in the form

$$\frac{dL}{d\tau} = \frac{\partial F}{\partial l}, \quad \frac{dG}{d\tau} = \frac{\partial F}{\partial g}, \quad \frac{dH}{d\tau} = \frac{\partial F}{\partial h} \tag{1.1}$$
$$\frac{dl}{d\tau} = -\frac{\partial F}{\partial L}, \quad \frac{dg}{d\tau} = -\frac{\partial F}{\partial G}, \quad \frac{dh}{d\tau} = -\frac{\partial F}{\partial H}$$
$$F = -\frac{G^2 - L^2}{2}\left(\sin^2 l + \frac{A}{B}\cos^2 l\right) - \frac{A}{2C}L^2 + U$$
$$U = \mu_i\,[(\alpha \sin \lambda - \beta \cos \lambda) \sin \varphi + \gamma \cos \varphi], \quad i = 1,\ 2$$

where either $\mu_1 = mrR/A$ or $\mu_2 = n^2mrR/n_2^{(0)2}A$. Obviously $0 < \mu_1 \ll 1$ if the fixed point O is fairly close to the body center of mass, and $0 < \mu_2 \ll 1$ provided that the angular velocity of rotation of the body is high in comparison with the angular velocity n (with $mrR/A \simeq 1$).

We introduce in the investigation the small parameter defined by $\mu = \max\{\mu_1, \ |A - B| / A\}$ on the supplementary assumption that density distribution in the body is close to axisymmetric and the parameters μ_1 and $|A - B| / A$ are of the same order. Let $\mu = \mu_1 = (q - 1)\delta^{-1}$, where $q = A / B$ and δ is a dimensionless parameter of order unity.

As the result, the Hamiltonian is reduce to the standard form

$$F = F_0(L, G) + \mu F_1(L, G, H, l, g) \tag{1.2}$$
$$F_0 = -G^2/2 - L^2(A/C - 1)/2$$
$$F_1 = f_{0,0} + f_{0,1}\cos g + f_{2,0}\cos 2l + f_{1,0}\cos(l + \lambda) +$$
$$f_{1,1}\cos(l + g + \lambda) + f_{1,-1}\cos(l - g + \lambda)$$

where the coefficients f_{k_1,k_2} are defined by the sequence of formulas

$$f_{0,0} = -{}^1/_4\delta G^2 \sin^2\theta + \cos\varphi\cos\rho\cos\theta, \quad f_{0,1} = -\cos\varphi\sin\rho\sin\theta \tag{1.3}$$
$$f_{2,0} = -{}^1/_4\delta G^2\sin^2\theta, \quad f_{1,0} = -\sin\varphi\cos\rho\sin\theta$$
$$f_{1,1} = -{}^1/_2\sin\varphi\sin\rho\,(1 + \cos\theta), \quad f_{1,-1} = -{}^1/_2\sin\rho\sin\varphi\,(-1 + \cos\theta)$$

The general solution of Eqs. (1.1) — (1.3) is defined by the initial values $L_0, G_0, H_0, l_0, g_0, h_0$ and the constant parameters $\varkappa = A/C,\ \delta,\ \varphi,\ \lambda,\ \mu$.

2. The next problem is to be establish the existence of periodic solutions of Eqs. (1.1) and to study these at small values of parameter μ.

When $\mu = 0$, the equations have a set of periodic solutions of period T

$$L = L_0, \quad G = G_0, \quad H = H_0 \tag{2.1}$$
$$l = n_1^{(0)}\tau + l_0, \quad g = n_2^{(0)}\tau + g_0, \quad h = h_0$$
$$n_1^{(0)} = (\varkappa - 1)L_0, \quad n_2^{(0)} = G_0; \quad G_0 = (\varkappa - 1)L_0N_1/N_2$$
$$N_1n_1^{(0)} = N_2n_2^{(0)}, \quad T = 2\pi N_1/n_2^{(0)} = 2\pi N_2/n_1^{(0)} \tag{2.2}$$

where N_1 and N_2 are integers (commensurability indicators), L_0, H_0, l_0, g_0, h_0 are arbitrary constants of integration, and G_0 is determined by the commensurability condition (2.2).

Solution (2.1), (2.2) implies that the dynamic symmetry axis $O\zeta$ of the body describes a cose of constant apex half-angle $\theta_0\,(\cos\theta_0 = L_0 / G_0)$ relative to the unperturbed moment of momentum vector $\mathbf{G}_0$ whose orientation in space is constant. Simultaneously the body uniformly rotates about its axis of symmetry at the angular velocity $n_1^{(0)}$. For every N_2 revolutions about the axis of inertia $O\zeta$ the body axis of symmetry describes the conical surface N_1 times.

For small values of parameter μ Eqs. (1.1) also admit periodic solutions of period T, but only in the case of those generating solutions of (2.1) that satisfy the group of conditions /3/

$$N_1n_1^{(0)} = N_2n_2^{(0)} \tag{2.3}$$
$$\Delta_1(F_0) \neq 0, \quad \Delta_2([F_1]) \neq 0 \tag{2.4}$$
$$d[F_1]/dg_0 = d[F_1]/dH_0 = 0 \tag{2.5}$$
$$[F_1] = \frac{1}{T}\int_0^T F_1(L_0, G_0, H_0, l_0, g_0, \tau)\,d\tau$$

where $[F_1]$ is function F_1 averaged over period T, $\Delta_{1,2}$ are Hessians of functions F_0 and F_1 with respect to respective variables L, G and g, H, calculated for the generating values of variables, i.e. for $\mu = 0$.

The condition of commensurability (2.3) determines the generating angle $\theta = \theta_0$ depending on the dynamic parameter $\varkappa = A/C$ and the type of commensurability $\cos\theta_0 = N_2/(N_1(\varkappa - 1))$ (Fig. 1).

Calculation shows that $\Delta_1(F_0) = \varkappa - 1$, which means that the first of conditions of existence (2.4) is satisfied for any generating solutions (2.1), (2.2).

If the commensurability indicator satisfies the inequality $|N_1| + |N_2| \geqslant 3$, then $[F_1] = f_{0,0}(L_0, G_0)$ and the second condition of existence in (2.4) is violated for the respective periodic solutions. In connection with this it is interesting to investigate periodic solutions to which correspond the commensurability indicators $N_1 = N_2 = 1$ or $N_1 = -N_2 = 1$.

In the case of commensurability $N_1 = N_2 = 1$ we have

$$[F_1] = f_{0,0}^{(0)} + f_{1,-1}^{(0)} \cos(l_0 - g_0 + \lambda)$$
$$f_{(0,0)}^{(0)} = -{}^1/_4 \delta G_0^2 \sin^2\theta_0 + \cos\varphi\cos\rho_0\cos\theta_0$$
$$f_{1,-1}^{(0)} = -{}^1/_2 \sin\rho_0 \sin\varphi\,(-1 + \cos\theta_0)$$

where the index (0) denotes the generating value of the respective variable.

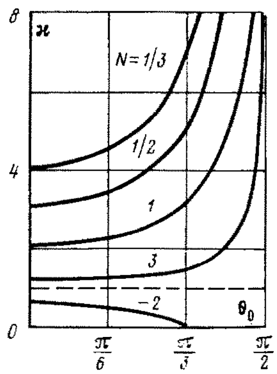

Fig.1

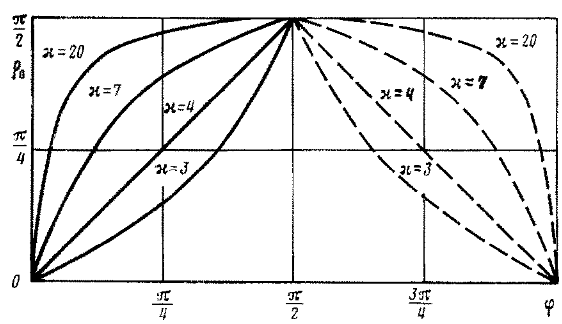

Fig.2

The first of Eqs.(2.5) has the following solutions:

$$1)\ \rho_0 = 0, \pi,\ 2)\ \theta_0 = 0,\ 3)\ \varphi = 0, \pi,\ 4)\ l_0 - g_0 + \lambda = 0, \pi,$$

and the second equation reduces to the form

$$\cos\varphi\sin\rho_0\cos\theta_0 + {}^1/_2\cos\rho_0\sin\varphi(-1 + \cos\theta_0)\cos(l_0 - g_0 + \lambda) = 0$$

The analysis of condition $\Delta_2 \neq 0$ shows that it is violated by solutions 1)—3). Hence the generating periodic solutions are determined by formulas

$$l_0 - g_0 + \lambda = 0, \pi,\ \cos\varphi\sin\rho_0\cos\theta_0 + {}^1/_2\cos\rho_0\sin\varphi\,(-1 + \cos\theta_0)\varepsilon = 0 \tag{2.6}$$
$$\varepsilon = \cos(l_0 - g_0 + \lambda) = \pm 1$$

for which

$$\Delta_2 = \frac{\sin\varphi\,(\cos\theta_0 - 1)\,\varepsilon}{2G_0^2\sin\rho_0}\left[\cos\varphi\cos\rho_0\cos\theta_0 + \frac{1}{2}\sin\rho_0\sin\varphi\,(\cos\theta_0 - 1)\,\varepsilon\right] \neq 0$$

except for $\varphi = 0, \pi/2, \pi$, $\theta_0 = 0, \pi/2$ ($\rho_0 = 0, \pi/2, \pi$).

Theorem 1. The equations of motion (1.1)—(1.3) of a heavy solid have, for fairly small values of parameter μ $(0 < \mu \leqslant \mu_0 \ll 1)$, a set of periodic solutions which are generated from solutions (2.1) of unperturbed equations for $\cos\theta_0 = (\varkappa - 1)^{-1}$, $\operatorname{tg}\rho_0 = \operatorname{tg}\varphi\,(\varkappa - 2)\varepsilon/2$, $l_0 - g_0 + \lambda = 0, \pi$ $(\theta_0 \neq 0, \pi/2;\ \varphi \neq 0, \pi/2, \pi)$, arbitrary initial conditions G_0, l_0, g_0, and parameters $\varkappa > 2$, $0 < \varphi < \pi$, $0 < \lambda \leqslant 2\pi$, δ, $0 < \mu \leqslant \mu_0 \ll 1$.

3. The equations of motion of a solid have two first integrals $F = c_1 = \text{const}$ and $H = c_2 = \text{const}$, because of this the four characteristic indices of the obtained periodic solutions are zero /3/. The two other characteristic indices can be formally represented in the form of series $\alpha^{(1,2)} = \alpha_1^{(1,2)}\sqrt{\mu} + \alpha_2^{(1,2)}\mu + \ldots$, whose basic coefficients are calculated using the method /3/

$$\alpha_1^{(1,2)} = \pm [{}^1/_2\, \varkappa \sin\rho_0 \sin\varphi\,(\cos\theta_0 - 1)\varepsilon]^{1/2}$$

Thus the necessary conditions of stability are satisfied by solutions (2.6) when $\varepsilon = 1$. The respective generating periodic solutions are determined by formulas

$$\cos\theta_0 = 1/(\varkappa - 1)\,(\varkappa > 2),\ \operatorname{tg}\rho_0 = \operatorname{tg}\varphi\,[(\varkappa - 2)/2],\ l_0 - g_0 + \lambda = 0 \tag{3.1}$$

Curves of $\rho_0(\varphi, \varkappa)$ corresponding to solution (3.1) are shown in Fig.2 by solid lines.

4. The periodic solutions of Eqs.(1.1)—(1.3) close to the generating solution (3.1) are represented by infinite series in powers of the small parameter μ

$$\begin{aligned} L &= L_0 + \mu L_1 + \mu^2 L_2 + \ldots \\ G &= G_0 + \mu G_1 + \mu^2 G_2 + \ldots \\ &\cdots\cdots\cdots\cdots \\ h &= h_0 + \mu h_1 + \mu^2 h_2 + \ldots \end{aligned} \tag{4.1}$$

where $L_s, G_s, \ldots, h_s\ (s = 1, 2, \ldots)$ are periodic functions of τ, which are to be determined. When $\mu = 0$, solution (4.1) becomes the respective generating periodic solution.

We present below the formulas for solution (4.1) accurate to μ

$$L = L_0 + \mu\,\{L_1^{(0)} + L_1^{(1)}\cos(\tau + l_0 + \lambda) + L_1^{(2)}\cos 2(\tau + l_0) + L_1^{(3)}\cos(2\tau + l_0 + g_0 + \lambda)\} \tag{4.2}$$

$$l = \tau + l_0 + \mu\,\{l_1^{(1)}\sin(\tau + l_0 + \lambda) + l_1^{(2)}\sin 2(\tau + l_0) + l_1^{(3)}\sin(\tau + g_0) + l_1^{(4)}\sin(2\tau + l_0 + g_0 + \lambda)\}$$

$$G = G_0 + \mu\,\{G_1^{(0)} + G_1^{(1)}\cos(\tau + g_0) + G_1^{(2)}\cos(2\tau + l_0 + g_0 + \lambda)\}$$

$$g = \tau + g_0 + \mu\,\{g_1^{(1)}\sin(\tau + l_0 + \lambda) + g_1^{(2)}\sin 2(\tau + l_0) + g_1^{(3)}\sin(\tau + g_0) + g_1^{(4)}\sin(2\tau + l_0 + g_0 + \lambda)\}$$

$$H = H_0 + \mu\,\{H_1^{(0)}\} \tag{4.3}$$

$$h = h_0 + \mu\,\{h_1^{(1)}\sin(\tau + l_0 + \lambda) + h_1^{(2)}\sin(\tau + g_0) + h_1^{(3)}\sin(2\tau + l_0 + g_0 + \lambda)\}$$

Coefficients $L_1^{(s)}, G_1^{(s)}, \ldots, h_1^{(s)}$ are determined using the sequence of formulas

$$\begin{aligned}
& L_1^{(0)} = (\varkappa - 1)^{-1}\left\{\frac{\delta}{2} G_0\cos\theta_0 + a\right\}, \quad G_1^{(0)} = -\frac{\delta}{2} G_0 + \cos\theta_0 a \\
& a = \frac{1}{G_0}\left(\cos\varphi\cos\rho_0 - \frac{1}{2}\sin\rho_0\sin\varphi\right) \\
& L_1^{(1)} = -\sin\varphi\cos\rho_0\sin\theta_0, \quad L_1^{(2)} = -\frac{\delta}{4} G_0^2\sin^2\theta_0, \\
& G_1^{(1)} = -\cos\varphi\sin\rho_0\sin\theta_0 \\
& G_1^{(2)} = L_1^{(3)} = -{}^1/_4\sin\varphi\sin\rho_0(1 + \cos\theta_0) \\
& l_1^{(1)} = -\frac{1}{G_0}\operatorname{ctg}\theta_0\sin\varphi\cos\rho_0 - (\varkappa - 1)\sin\varphi\cos\rho_0\sin\theta_0, \\
& l_1^{(3)} = -\frac{1}{G_0}\operatorname{ctg}\theta_0\cos\varphi\sin\rho_0 \\
& l_1^{(2)} = \frac{\delta}{4} G_0\cos\theta_0 - \frac{\delta}{8}(\varkappa - 1) G_0^2\sin^2\theta_0, \\
& l_1^{(4)} = \frac{1}{4G_0}\sin\varphi\sin\rho_0 + \frac{1}{2}(\varkappa - 1) G_1^{(2)} \\
& g_1^{(1)} = \frac{\sin\varphi\cos\rho_0\cos 2\theta_0}{G_0\sin\theta_0}, \quad g_1^{(2)} = \frac{\delta G_0}{4} \\
& g_1^{(3)} = -\frac{\cos\varphi}{G_0\sin\rho_0\sin\varphi}(\cos^2\rho_0\sin^2\theta_0 + \cos^2\theta_0\sin^2\rho_0) + G_1^{(1)} \\
& g_1^{(4)} = -\frac{\sin\varphi}{4G_0}[\operatorname{ctg}\rho_0\cos\rho_0(1 + \cos\theta_0) - \cos\theta_0\sin\rho_0] + \frac{1}{2} G_1^{(2)} \\
& h_1^{(1)} = -\frac{L_1^{(1)}}{G_0}, \quad h_1^{(2)} = -\frac{1}{G_0}\operatorname{ctg}\rho_0\cos\rho_0\cos\varphi\sin\theta_0 \\
& h_1^{(3)} = -\frac{1}{4G_0}\operatorname{ctg}\rho_0\cos\rho_0\sin\varphi(1 + \cos\theta_0)
\end{aligned} \tag{4.4}$$

The expression for $H_1^{(0)}$ is not adduced owing to its unwieldiness.

Note that the coefficients in (4.4) are calculated for generating values of $G_0, \theta_0, \rho_0, l_0, g_0$, determined by formulas (3.1).

Let us list some of the properties of periodic solutions, viz:
1) the initial conditions of these solutions are approximately defined by formulas

$$L = L_0 + \mu L_1^{(0)} + \ldots, \quad G = G_0 + \mu G_1^{(0)} + \ldots,$$
$$h = h_0 + \ldots$$

2) projection of the moment of momentum vector on the Oz axis is constant and is $H_0 + \mu H_1^{(0)} + \ldots = \mathrm{const}$ for each periodic solution of (4.2)— (4.4);
3) the moment of momentum vector $\mathbf{G}$ performs periodic oscillations of small amplitude close to its initial position in space;
4) small amplitude periodic oscillations determined by formulas (4.4) are superposed on the regular precession motion of the body relative to vector $\mathbf{G}$.

5. We now use the first integral $H = c_2$ to lower the order of Eqs. (1.1) by two units, reducing by this the problem to a two-stage one and the equations of motion to the form

$$\frac{dL}{d\tau} = \frac{\partial F}{\partial l}, \quad \frac{dl}{d\tau} = -\frac{\partial F}{\partial L}, \quad \frac{dG}{d\tau} = \frac{\partial F}{\partial g}, \quad \frac{dg}{\partial \tau} = -\frac{\partial F}{\partial G} \tag{5.1}$$

The Hamiltonian F is determined using the same (1.2) and (1.3) formulas in which it is sufficient to set $\cos\rho = c_2/G$, i.e. $F = F_0(L, G) + \mu F_1(L, G, l, g)$. Having obtained any solution of Eqs. (5.1), we calculate variables H and h by formulas

$$H = c_2, \quad h - h_0 = -\int_{\tau_0}^{\tau} \frac{\partial F}{\partial H} d\tau \quad (h_0 = h(\tau_0)) \tag{5.2}$$

Existence of Poincaré periodic solutions for Eqs. (5.1) was investigated in /1/.

Equations (5.1) have a set of periodic solutions which for $\mu = 0$ become the following generating solutions:

$$L = L_0, \; G = G_0, \; l = n_1^{(0)}\tau + l_0, \; g = n_2^{(0)}\tau + g_0 \tag{5.3}$$
$$n_1^{(0)} = n_2^{(0)} = (\varkappa - 1)L_0 = G_0, \; l_0 - g_0 + \lambda = 0, \; \pi, \; \cos\theta_0 = (\varkappa - 1)^{-1} \, (\varkappa > 2)$$

which is periodic of period $T = 2\pi/G_0$ and corresponds to the case of commensurability $n_1^{(0)} = n_2^{(0)}$ (similar solutions obtain in the case of commensurability $n_1^{(0)} = -n_2^{(0)}$).

In the generating solution the following are selected arbitrarily: G_0, the magnitude of vector $\mathbf{G}$ which determines the period of periodic solution; g_0, the angular distance between the planes Oxy and $O\xi\eta$ on the intermediate plane; and the constant parameter $\varkappa > 2$. Solutions to which correspond the following values $c_2 = 0$, $\varphi = 0, \pi/2, \pi$, $\theta_0 = 0$ are to be excluded from solutions (5.3).

Principal terms of series representing periodic solutions are defined by formulas (4.2) and (4.4) in which $\rho_0, \varphi, G_0, \varkappa, \delta$, are generally arbitrary.

The necessary conditions of stability are satisfied by those of periodic solutions for which $\cos\theta_0 = (\varkappa - 1)^{-1}$, $l_0 - g_0 + \lambda = 0$.

Generally conditionally periodic solutions of the input equations (4.1) correspond to periodic solutions of the reduced system of Eqs. (5.1).

Indeed, by calculating the quadrature in (5.2) using L, G, l, g defined by formulas (4.2) and (4.4) we obtain

$$h - h_0 = \mu\Lambda\tau + \mu\,\{(h_1^{(1)} + h_1^{(2)})\sin(\tau + g_0) + h_1^{(3)}\sin 2(\tau + g_0)\}$$

where $h_1^{(1)}, h_1^{(2)}, h_1^{(3)}$ are constant coefficients determined by formulas (4.4) with the generating values of variables G_0, θ_0 and arbitrary angular velocity ρ_0, a $\mu\Lambda n$ of the "secular" variation of the position of the moment of momentum vector $\mathbf{G}$

$$\Lambda = G_0^{-1}\operatorname{cosec}\rho_0\,[\cos\varphi\cos\theta_0 + {}^1/_2\sin\varphi\operatorname{ctg}\rho_0\,(1 + \cos\theta_0)\,\varepsilon]$$

Thus in the case of the considered here solutions vector $\mathbf{G}$ describes during time $T_h = 2\pi/(\mu\Lambda n)$ a cone with the apex half-angle ρ about a vertical line. At the same time oscillations of small amplitude of order μ are superposed on the slow uniform rotation of vector $\mathbf{G}$. In the coordinate system attached to vector $\mathbf{G}$ a body which is close to a dynamically axisymmetric performs periodic rotary motions of period $T = 2\pi/n_1^{(0)}$. In the particular case, when $\Lambda = 0$, vector $\mathbf{G}$ performs only periodic oscillations of small amplitude about its initial position. Then ρ_0 is determined using formula (2.6) and the conditionally periodic solution becomes the periodic solution (4.2)— (4.4).

Theorem 2. With fairly small values of parameter μ the equations of motion of a heavy solid about a fixed point admit a 9-parameter set of conditionally periodic solutions for which the initial conditions G_0, g_0, ρ_0, h_0 and the problem parameters $\varkappa > 2$, $0 < \varphi < \pi$ $(\varphi \neq \pi/2)$, $0 \leqslant \lambda \leqslant 2\pi$, δ, $0 < \mu \leqslant \mu_0 \ll 1$. are aribtrarily selected.

Note that the obtained here solutions are of a very general character, since they contain 9 arbitrary parameters. For comparison we present the respective arbitrary initial conditions and dimensionless constants in known solutions of this problem.

The Euler solution: 8 parameters $G_0, \theta_0, \rho_0, l_0, g_0, h_0, \varkappa, q$, constraint $r=0$.

The Lagrange solution: 9 parameters $G_0, \theta_0, \rho_0, l_0, g_0, h_0, \varkappa, r, \lambda$, constraints on parameters $q=1$, $\varphi=0, \pi$.

The Kovalevska solution: 8 parameters $G_0, \theta_0, \rho_0, l_0, g_0, h_0, r, \lambda$, constraints on parameters $\varkappa=2$, $q=1$, $\varphi=\pi/2$.

The Hesse— Appelroth solution: 8 parameters $G_0, \rho_0, l_0, g_0, h_0, \varkappa, q, r$, constraints on initial conditions and parameters

$$\operatorname{tg}\theta_0\sin l_0=\mp\sqrt{\frac{\varkappa-q}{q-1}}\varepsilon_0,\quad \lambda=\frac{\pi}{2};\ \frac{3}{2}\pi,\quad \operatorname{tg}\varphi=\pm\sqrt{\frac{q-1}{\varkappa-q}},\quad \varepsilon_0=\pm1$$

6. Using the energy integral $F=c_1$ we lower the order of Eqs.(5.1) by two units and reduce them to the form of Whittaker equations. For this we apply to equation

$$F_0(L,G)+\mu F_1(L,G,l,g)-c_1=0$$

the theorem on implicit function and solve it for the variable G. We obtain

$$G=\Phi=\Phi_0+\mu\Phi_1+\mu^2\Phi_2+\ldots,\quad \Phi_0=[-2c_1-(\varkappa-1)L^2]^{1/2},\quad \Phi_1=-\frac{F_1}{\Phi_0},\quad \Phi_2=-F_1\left(\frac{\partial F_1}{\partial G}\right)_{G=\Phi_0}-\frac{F_1^2}{2\Phi_0}$$

This enables us to write the equations of motion of a solid in the form

$$\frac{dL}{dg}=\frac{\partial\Phi}{\partial l},\quad \frac{dl}{dg}=-\frac{\partial\Phi}{\partial L} \tag{6.1}$$
$$\Phi=\Phi_0(L)+\mu\Phi_1(L,l,g)+\mu^2\Phi_2(L,l,g)+\ldots$$

where Φ is the new Hamiltonian representable in the form of infinite series in powers of parameter μ, and convergent for fairly small values of that parameter $0<\mu\leqslant\mu_0\ll1$.

If one succeeds in obtaining some solution of Eqs.(6.1), a solution of the input differential equations (1.1) defined in quadratures

$$H=c_2,\quad G=\Phi(L,l,g) \tag{6.2}$$
$$\int\limits_{g_0}^{g}\frac{dg}{\partial F/\partial G}=\tau_0-\tau,\quad \int\limits_{\tau_0}^{\tau}\frac{\partial F}{\partial H}d\tau=h_0-h \tag{6.3}$$

where g_0, h_0 are values of variables g and h when $\tau=\tau_0$, corresponds to that solution.

When $\mu=0$ Eqs.(6.1) have periodic solutions

$$L=L_0,\quad l=l^0=n^{(0)}g+l_0,\quad n^{(0)}=\partial\Phi_0/\partial L_0=(\varkappa-1)L_0/\Phi_0=n_1^{(0)}/n_2^{(0)}=N_2/N_1 \tag{6.4}$$

where $n^{(0)}$ is a rational number.

We seek periodic solutions of Eqs.(6.1) close to solutions (6.4) in the form of power series

$$L=L_0+\mu L_1+\mu^2L_2+\ldots,\quad l=l^0+\mu l_1+\mu^2l_2+\ldots \tag{6.5}$$

where L_s, l_s are periodic functions of variable g.

In the case of commensurability $N_1=N_2=1$, considered above for the input differential equations, the conditions of existence of series (6.5) are of the simple form

$$\partial^2\Phi_0/\partial L_0^2\neq0,\quad \partial[\Phi_1]/\partial l_0=0,\quad \partial^2[\Phi_1]/\partial l_0^2\neq0$$
$$[\Phi_1]=-G_0^{-1}[f_{0,0}(L_0)+f_{1,-1}(L_0)\cos(l_0+\lambda)]$$
$$f_{0,0}={}^1\!/_4\delta G_0^2\sin^2\theta_0-\cos\varphi\cos\rho_0\cos\theta_0,$$
$$f_{1,-1}={}^1\!/_2\sin\rho_0\sin\varphi(-1+\cos\theta_0)$$
$$G_0=[-2c_1-(\varkappa-1)L_0^2]^{1/2},\quad \cos\rho_0=c_2/G_0,\quad \cos\theta_0=L_0/G_0$$

where $[\Phi_1]$ represents function $\Phi_1=\Phi_1(L_0,l_0,g)$. averaged over the period $T=2\pi$.

Thus solutions of Eqs.(6.1) periodic in g exist when parameter μ is reasonably small and $L_0=\pm[2c_1/(\varkappa(1-\varkappa))]^{1/2}$, $l_2+\lambda=0,\pi$, if only $\rho_0\neq0,\pi$; $\varphi\neq0,\pi$; $\theta_0\neq0$.

The solutions of input equations (1.1) correspond to the obtained here periodic solutions. Without attempting to derive these solutions, we shall point out their characteristic singularities.

The first of Eqs.(6.3) enables us to determine function $g(\tau)$. We denote by $\bar n=-[\partial F/\partial G_0]$

the constant constituent $\partial F/\partial G$ of functions in relation to variable g, and transform this equation to the form of Lagrange equation

$$g = \sigma + \mu\psi(g, \mu) \qquad (6.6)$$
$$\sigma = \bar{n}(\tau - \tau_0), \quad \bar{n} = -\frac{\partial F_0}{\partial G_0} - \mu\frac{\partial [F_1]}{\partial G_0}$$

where $[F_1]$ means the averaging of function F_1 with respect to variable g.

The solution of Eq.(6.6) is represented by a Lagrange series. Consequently, the remainder $g - \sigma$ is a periodic function of variable τ of period $\bar{T} = 2\pi/\bar{n}$ which differs from the period of the respective generating solution $T = 2\pi/n$. This shows that the variables L, l, G, g are in conformity with formulas (6.2) and (6.3) also periodic functions of τ of period $\bar{T}$. Thus the periods of the generating and respective periodic solutions differ by a quantity of order μ. In celestial mechanics similar solutions are called periodic solutions of the Schwarzschild type.

Finally, the second quadrature in (6.3) determines the variable $h(\tau)$ as a conditionally periodic function of the form $h(\tau) = \mu\Lambda\tau + \Psi(\tau)$, where $\mu\Lambda n$ is the angular precession velocity of vector $\mathbf{G}$, and $\Psi(\tau)$ is a periodic function of τ of period $\bar{T}$.

We note in conclusion that similar classes of solutions can be obtained and investigated by the same method in the case of a solid with a dynamically arbitrary structure on the assumption that $\mu = (n/n_1^{(0)})^2 \ll 1$. For this it is sufficient to use equations of motion in terms of variables "action—angle" introduced on the basis of the Euler—Poinsot problem. New sets of periodic and conditionally periodic solutions and, moreover, be readily established by using the method of Poincaré and of equations of rotary motion of a solid, expressed in terms of the "action—angle" variables that are to be obtained on the basis of problems of solid body dynamics with the Lagrange and Kovalevska solutions.

REFERENCES

1. KOSLOV V.V., New periodic solutions for the problem of motion of a heavy solid body around a fixed point. PMM, Vol.39, No.3, 1975.
2. DEMIN V.G. and KISELEV F.I., On periodic motions of a rigid body in a central Newtonian field. PMM Vol.38, No.2, 1974.
3. POINCARÉ A., Selected Works, Vol.1 /Russian translation/. Moscow, NAUKA, 1971.

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